

Ward identities for particle-particle and particle-hole pairs *

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Abstract

Ward identities for charge-density and spin-density fluctuations are discussed in comparison with those for superconducting fluctuations.

1 Introduction

Recently I have reported the discussion of Ward identities for superconducting fluctuations [1, 2]. In this note I will extend it to the case of charge-density and spin-density fluctuations. While the superconducting fluctuation is described by particle-particle pair propagator, the charge-density and spin-density fluctuations are described by particle-hole pair propagator.

For the sake of clarity I will only discuss the case of zero temperature. At finite temperature we can use the same Ward identity with thermal frequency [1]. In addition I will only discuss the local pairs for simplicity. The discussion of nonlocal pairs has been already given in ref. [2].

Following description is based on refs. [1] and [2].

2 Ward identity for electric current vertex

Under the charge-conservation law

$$\sum_{\mu=0}^3 \frac{\partial}{\partial z_{\mu}} j_{\mu}^e(z) = 0, \quad (1)$$

*In v2 eqs. (24) and (25) have been added and footnote-2 has been corrected.

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the three-point function $M_\mu^e(x, y, z)$ shown in Fig. 1

$$M_\mu^e(x, y, z) = \langle T \{ j_\mu^e(z) A(x) A^\dagger(y) \} \rangle, \quad (2)$$

satisfies the relation

$$\begin{aligned} \sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} M_\mu^e(x, y, z) = & \left\langle T \left\{ [j_0^e(z), A(x)] A^\dagger(y) \right\} \right\rangle \delta(z_0 - x_0) \\ & + \left\langle T \left\{ A(x) [j_0^e(z), A^\dagger(y)] \right\} \right\rangle \delta(z_0 - y_0), \end{aligned} \quad (3)$$

where

$$j_0^e(z) = e\psi_\uparrow^\dagger(z)\psi_\uparrow(z) + e\psi_\downarrow^\dagger(z)\psi_\downarrow(z), \quad (4)$$

and $A(x)$ and $A^\dagger(y)$ represent the annihilation and creation operators of particle-particle pairs

$$\Psi^\dagger(x) \equiv \psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x), \quad \Psi(x) \equiv \psi_\downarrow(x)\psi_\uparrow(x), \quad (5)$$

or particle-hole pairs

$$\rho_\uparrow(x) \equiv \psi_\uparrow^\dagger(x)\psi_\uparrow(x), \quad \rho_\downarrow(x) \equiv \psi_\downarrow^\dagger(x)\psi_\downarrow(x), \quad (6)$$

$$\sigma_+(x) \equiv \psi_\uparrow^\dagger(x)\psi_\downarrow(x), \quad \sigma_-(x) \equiv \psi_\downarrow^\dagger(x)\psi_\uparrow(x). \quad (7)$$

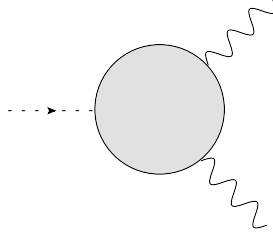


Figure 1: Feynman diagram for three-point function M_μ^e : The shaded circle represents the coupling of the external electromagnetic field (broken line) and the particle-particle or particle-hole fluctuations (wavy lines). All the Feynman diagrams in this note are drawn by JaxoDraw [3].

Using the commutation relations for particle-particle pairs

$$[j_0^e(z), A(x)]\delta(z_0 - x_0) = -2eA(x)\delta^4(z - x), \quad (8)$$

$$[j_0^e(z), A^\dagger(y)]\delta(z_0 - y_0) = 2eA^\dagger(y)\delta^4(z - y), \quad (9)$$

we have obtained the Ward identity for particle-particle pairs [1, 2]. On the other hand, the commutation relations for particle-hole pairs

$$[j_0^e(z), A(x)]\delta(z_0 - x_0) = 0, \quad (10)$$

$$[j_0^e(z), A^\dagger(y)]\delta(z_0 - y_0) = 0, \quad (11)$$

with $j_0^e(z) = e(\rho_\uparrow(z) + \rho_\downarrow(z))$ lead the Ward identity for the electric current vertex of particle-hole pairs

$$\sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} M_\mu^e(x, y, z) = 0. \quad (12)$$

It should be noted here that the commutation relation picks up the charge¹ carried by the pair [4]. Particle-hole pairs are charge-neutral and carry no charge. Namely charge- and spin-density fluctuations do not couple to electromagnetic field.

The relation, eq. (3), is Fourier transformed into

$$\begin{aligned} \sum_{\mu=0}^3 ik_\mu M_\mu^e(q, q - k) &= \int d(x_0 - y_0) e^{-iq_0(x_0 - y_0)} \int d(z_0 - x_0) e^{-ik_0(z_0 - x_0)} \\ &\times \left(\left\langle T \left\{ [j_k^e(x_0), A_{\vec{q}-\vec{k}}(x_0)] A_{\vec{q}}^\dagger(y_0) \right\} \right\rangle \delta(z_0 - x_0) \right. \\ &\quad \left. + \left\langle T \left\{ A_{\vec{q}-\vec{k}}(x_0) [j_k^e(y_0), A_{\vec{q}}^\dagger(y_0)] \right\} \right\rangle \delta(z_0 - y_0) \right). \end{aligned} \quad (13)$$

Using the commutation relations for particle-particle pairs

$$[j_k^e, A_{\vec{q}-\vec{k}}] = -2eA_{\vec{q}}, \quad [j_k^e, A_{\vec{q}}^\dagger] = 2eA_{\vec{q}-\vec{k}}^\dagger, \quad (14)$$

we have obtained the Ward identity for particle-particle pairs [1, 2]. On the other hand, the commutation relations for particle-hole pairs

$$[j_k^e, A_{\vec{q}-\vec{k}}] = 0, \quad [j_k^e, A_{\vec{q}}^\dagger] = 0, \quad (15)$$

lead the Ward identity for the electric current vertex of particle-hole pairs

$$\sum_{\mu=0}^3 k_\mu M_\mu^e(q, q - k) = 0. \quad (16)$$

¹The electric charge \mathcal{Q} is given as $\mathcal{Q} = \int d\vec{z} j_0^e(z)$. Here we consider the charge carried by local pairs.

As has been discussed in ref. [1] the current vertex for dc conductivity is obtained from the Ward identity in the limit of vanishing external momentum $k \rightarrow 0$. Thus the Ward identity, eq. (16), means that the current vertex of particle-hole pairs has no contribution to dc conductivity. Such a conclusion is natural, since particle-hole pairs are charge-neutral and carry no charge. A perturbational example of “no contribution” is shown in Fig. 2.

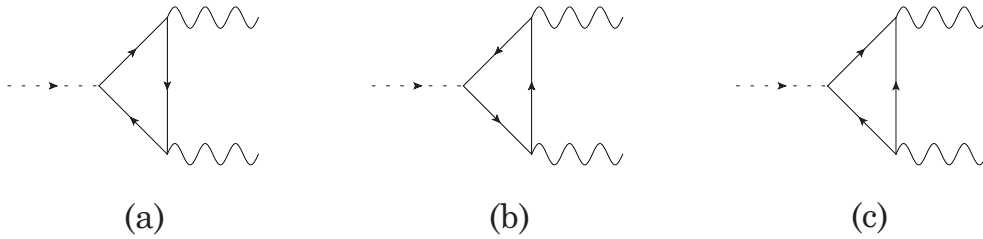


Figure 2: Feynman diagrams for Aslamazov-Larkin processes: Wavy lines represent particle-hole fluctuation in (a) and (b) and particle-particle fluctuation in (c). Solid lines represents electron propagator. The contributions of (a) and (b) cancel and the resulting electric current vertex for particle-hole pairs vanishes in the case of dc conductivity. Such a cancelation has been discussed in ref. [5] for spin-density fluctuation and in ref. [6] for charge-density fluctuation. In the case of particle-particle pairs there is no canceling partner when three lines coupling to the triangle are fixed. Then superconducting fluctuation couples to external electromagnetic field.

3 Ward identity for more complex vertex

The Ward identity discussed in the previous section is a special case of the multi-point function $M_\mu^{(n,p)}$

$$M_\mu^{(n,p)} = \left\langle T \left\{ j_\mu^e(z) \prod_{i=1}^n \psi(x_i) \psi^\dagger(y_i) \prod_{j=1}^p A_j(z_j) \right\} \right\rangle, \quad (17)$$

shown in Fig. 3.

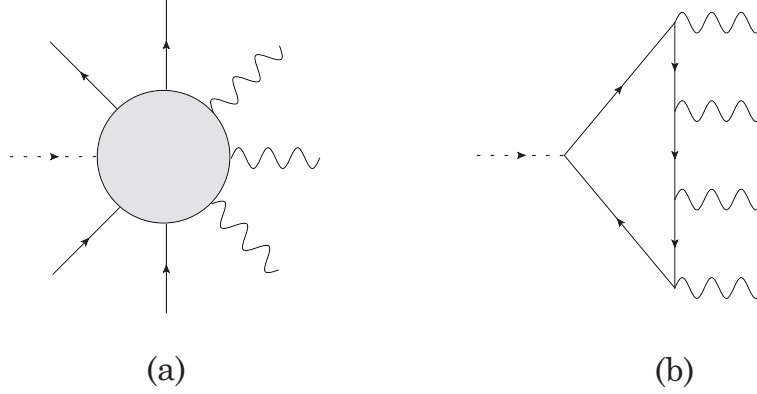


Figure 3: Feynman diagrams for multi-point function $M_\mu^{(n,p)}$: (a) A general expression for $n = 2$ and $p = 3$. (b) A perturbational expression for $n = 0$ and $p = 4$. Perturbationally “no contribution” of this type has been discussed in ref. [6].

The following derivation of the Ward identity is given in ref. [4] for QED. Here the spin-index of electron operator is dropped and the distinction between creation and annihilation operators of pairs is neglected. The derivative of the time-ordering operator T leads the identity

$$\begin{aligned} \frac{\partial}{\partial t} T \left\{ \hat{O}(t) \hat{O}_1(t_1) \hat{O}_2(t_2) \cdots \hat{O}_n(t_n) \right\} &= T \left\{ \frac{\partial \hat{O}(t)}{\partial t} \hat{O}_1(t_1) \hat{O}_2(t_2) \cdots \hat{O}_n(t_n) \right\} \\ &+ \sum_{i=1}^n T \left\{ \hat{O}_1(t_1) \cdots \hat{O}_{i-1}(t_{i-1}) [\hat{O}(t), \hat{O}_i(t_i)] \hat{O}_{i+1}(t_{i+1}) \cdots \hat{O}_n(t_n) \right\} \delta(t - t_i), \end{aligned} \quad (18)$$

where \hat{O} and \hat{O}_i ($i = 1, 2, \dots, n$) represent arbitrary operators. Making use of this identity the divergence of $M_\mu^{(n,p)}$ is shown to satisfy the relation²

$$\begin{aligned}
& \sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} M_\mu^{(n,p)} \\
&= \sum_{i=1}^n \left\langle T \left\{ \psi(x_1) \psi^\dagger(y_1) \cdots \psi(x_{i-1}) \psi^\dagger(y_{i-1}) \right. \right. \\
&\quad \times \left. \left[\psi(x_i) \psi^\dagger(y_i) \right] \psi(x_{i+1}) \psi^\dagger(y_{i+1}) \cdots \psi(x_n) \psi^\dagger(y_n) \prod_{j=1}^p A_j(z_j) \right\} \right\rangle \\
&+ \sum_{j=1}^p \left\langle T \left\{ \prod_{i=1}^n \psi(x_i) \psi^\dagger(y_i) A_1(z_1) \cdots A_{j-1}(z_{j-1}) \left[A_j(z_j) \right] A_{j+1}(z_{j+1}) \cdots A_p(z_p) \right\} \right\rangle,
\end{aligned} \tag{19}$$

under the charge-conservation law. The commutation relations for electrons result in

$$[j_0^e(z), \psi(x)] \delta(z_0 - x_0) = -e \psi(x) \delta^4(z - x), \tag{20}$$

$$[j_0^e(z), \psi^\dagger(y)] \delta(z_0 - y_0) = e \psi^\dagger(y) \delta^4(z - y). \tag{21}$$

If we consider only particle-hole pairs, then the commutation relation for them results in

$$[j_0^e(z), A_j(z')] \delta(z_0 - z'_0) = 0, \tag{22}$$

where A_j is a linear combination of $\{\rho_\uparrow, \rho_\downarrow, \sigma_+, \sigma_-\}$. Using these commutation relations we obtain the Ward identity

$$\sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} M_\mu^{(n,p)} = e \left\langle T \left\{ \prod_{i=1}^n \psi(x_i) \psi^\dagger(y_i) \prod_{j=1}^p A_j(z_j) \right\} \right\rangle \sum_{i=1}^n \left(\delta^4(z - y_i) - \delta^4(z - x_i) \right). \tag{23}$$

This Ward identity is identical to that in QED.

An application of this Ward identity is the proof of “no contribution” of multi-fluctuations of particle-hole pairs shown in Fig. 3.

² Here we have introduced the notation

$$\begin{aligned}
[\psi(x) \psi^\dagger(y)] &\equiv [j_0^e(z), \psi(x)] \psi^\dagger(y) \delta(z_0 - x_0) + \psi(x) [j_0^e(z), \psi^\dagger(y)] \delta(z_0 - y_0), \\
[A_j(z')] &\equiv [j_0^e(z), A_j(z')] \delta(z_0 - z'_0).
\end{aligned}$$

If we consider the multi-point function $M_\mu^{(n,m)}$

$$M_\mu^{(n,m)} = \left\langle T \left\{ j_\mu^e(z) \prod_{i=1}^n \psi(x_i) \psi^\dagger(y_i) \prod_{j=1}^m A(x_j) A^\dagger(y_j) \right\} \right\rangle, \quad (24)$$

where $A(x_j)$ and $A^\dagger(y_j)$ are the annihilation and creation operators of particle-particle pairs, we obtain the Ward identity

$$\begin{aligned} \sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} M_\mu^{(n,m)} &= e \left\langle T \left\{ \prod_{i=1}^n \psi(x_i) \psi^\dagger(y_i) \prod_{j=1}^m A(x_j) A^\dagger(y_j) \right\} \right\rangle \sum_{i=1}^n \left(\delta^4(z - y_i) - \delta^4(z - x_i) \right) \\ &\quad + 2e \left\langle T \left\{ \prod_{i=1}^n \psi(x_i) \psi^\dagger(y_i) \prod_{j=1}^m A(x_j) A^\dagger(y_j) \right\} \right\rangle \sum_{j=1}^m \left(\delta^4(z - y_j) - \delta^4(z - x_j) \right). \end{aligned} \quad (25)$$

4 Ward identity for heat current vertex

Under the energy-conservation law

$$\sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} j_\mu^Q(z) = 0, \quad (26)$$

the three-point function $M_\mu^Q(x, y, z)$

$$M_\mu^Q(x, y, z) = \left\langle T \{ j_\mu^Q(z) A(x) A^\dagger(y) \} \right\rangle, \quad (27)$$

satisfies the relation

$$\begin{aligned} \sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} M_\mu^Q(x, y, z) &= \left\langle T \left\{ [j_0^Q(z), A(x)] A^\dagger(y) \right\} \right\rangle \delta(z_0 - x_0) \\ &\quad + \left\langle T \left\{ A(x) [j_0^Q(z), A^\dagger(y)] \right\} \right\rangle \delta(z_0 - y_0). \end{aligned} \quad (28)$$

If the interaction among electrons is local, then using³

$$[j_0^Q(x), A(x)] = [H, A(x)], \quad [j_0^Q(y), A^\dagger(y)] = [H, A^\dagger(y)], \quad (29)$$

we easily obtain the Ward identity for heat current vertex as discussed in ref. [1]. Even if the interaction is nonlocal, we can obtain the same Ward

³ Here $j_0^Q(z)$ is the energy density and the Hamiltonian H is given as $H = \int d\vec{z} j_0^Q(z)$.

identity employing the Fourier transform

$$\begin{aligned} \sum_{\mu=0}^3 ik_{\mu} M_{\mu}^e(q, q-k) &= \int d(x_0 - y_0) e^{-iq_0(x_0 - y_0)} \int d(z_0 - x_0) e^{-ik_0(z_0 - x_0)} \\ &\times \left(\left\langle T \left\{ [j_{\vec{k}}^Q(x_0), A_{\vec{q}-\vec{k}}(x_0)] A_{\vec{q}}^{\dagger}(y_0) \right\} \right\rangle \delta(z_0 - x_0) \right. \\ &\quad \left. + \left\langle T \left\{ A_{\vec{q}-\vec{k}}(x_0) [j_{\vec{k}}^Q(y_0), A_{\vec{q}}^{\dagger}(y_0)] \right\} \right\rangle \delta(z_0 - y_0) \right). \end{aligned} \quad (30)$$

Here we can make the replacement

$$[j_{\vec{k}}^Q, A_{\vec{q}-\vec{k}}] \Rightarrow [H, A_{\vec{q}}], \quad [j_{\vec{k}}^Q, A_{\vec{q}}^{\dagger}] \Rightarrow [H, A_{\vec{q}-\vec{k}}^{\dagger}], \quad (31)$$

in the limit of vanishing external momentum, $\vec{k} \rightarrow 0$.

Irrespective of the range of the interaction we obtain the Ward identity

$$\sum_{\mu=0}^3 k_{\mu} M_{\mu}^Q(q+k, q) = (q_0 + k_0) D(q+k) - q_0 D(q), \quad (32)$$

for any pairs. This result is consistent with the Ward identity for phonons [7]. Here $D(q)$ is the Fourier transform of the fluctuation propagator

$$D(x, y) = -i \langle T \{ A(x) A^{\dagger}(y) \} \rangle, \quad (33)$$

for particle-particle or particle-hole pairs. The pair with four-momentum q carries the energy of q_0 .

5 Conclusion

In the discussion of the Ward identity the commutation relation plays the central role.

In the case of the electric current vertex it picks up the charge of the pair. Thus it is concluded that charge-neutral pairs do not couple to electromagnetic field. Namely charge- and spin-density fluctuations do not carry charge. On the other hand, Cooper pairs carrying charge $2e$ couple to electromagnetic field.

In the case of the heat current vertex it picks up the energy of the pair.

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⁴See v2 with APPENDIX.

⁵See §8-4-1.